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Some Classes of Generating Functions for the Laguerre and Hermite Polynomials

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Abstract. In the first half of the article, we present two theorems which give, as special cases, a number of new classes of generating functions for the Laguerre polynomial. These formulae extend the recent results of Carlitz [2] and others. The latter part of our work deals with two theorems involving new generating functions for the Hermite and generalized Hermite polynomials, thus generalizing some well-known expansions. The method of proof adopted in this paper differs from that of previous workers.

I. Introduction. Extending the work done by Chatterjea [3] and Brown [1], Carlitz [2] gave the generating function

$$\sum_{n=0}^{\infty} t^n L_n^{a+bn}(x) = \frac{(1+v)^{a+1}}{1-bv} \exp(-xv), \tag{1.1}$$

where v satisfies $v = t(1+v)^{b+1}$, $v(0) = 0$, and a, b are arbitrary complex numbers. The Laguerre polynomial is defined in the usual way [6, Volume II, p. 188].

In our presentation, we derive two theorems which give expansions for the Laguerre polynomial. A special case of Theorem 1 is seen to be

$$\begin{aligned}&\sum_{n=0}^{\infty} t^n L_n^{v+bn}[x(1+an)] \\ &\quad = (1-z)^{1-v}[1-z(b+2-ax)+z^2(b+1)]^{-1}\exp[xz/(z-1)],\end{aligned} \tag{1.2}$$

where $t = z(1-z)^b \exp[axz/(1-z)]$, v, a, b are arbitrary complex numbers and $|t| < 1$. Letting $a = 0$ in (1.2) gives essentially (1.1).

Theorem 2 yields a number of new expressions. One class of generating functions is

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{t^n}{(v+bn+n)} L_n^{v+bn}[x(1+an)] \\ &\quad = (v)^{-1}(1-z)^{-v}\exp[xz/(z-1)]\, {}_1F_1\left[\begin{matrix} 1; \\ \dfrac{v+1+b}{1+b}; \end{matrix} \quad \frac{xz(1+b-av)}{(1-z)(1+b)}\right],\end{aligned} \tag{1.3}$$

where $t = z(1-z)^b \exp[axz/(1-z)]$ and $|t| < 1$.

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The right-hand side of (1.3) is the incomplete gamma function. See Chapter IV of Luke [9] for extensive treatment and bibliography of this function. A reduction of interest involves the case of $1 + b = av$, and we obtain

$$(1.4)\qquad \sum_{n=0}^{\infty} \frac{t^n}{(1+an)} L_n^{v+avn-n}[x(1+an)] = (1-z)^{-v}\exp[xz/(z-1)],$$

where $t = z(1-z)^{av-1}\exp[axz/(1-z)]$ and $|t| < 1$. For $a = 0$ in (1.3), we deduce the known result and its extension given by Eq. (4.13) of [4].

A second class of generating functions, a consequence of Theorem 2, may be expressed as

$$(1.5)\qquad \begin{aligned} &\sum_{n=0}^{\infty} \frac{t^n}{(1+an)} L_n^{v+bn}[x(1+an)] \\ &= (1-z)^{-v}\exp[xz/(z-1)]\,{}_2F_1\left[\begin{matrix} \dfrac{1+b-av}{a}, & 1; \\ & z \\ \dfrac{a+1}{a}; & \end{matrix}\right] \end{aligned}$$

where $t = z(1-z)^b\exp[axz/(1-z)]$ and $|t| < 1$. The right-hand side of (1.5) is the incomplete beta function [9, p. 299]. Putting $b = a + av$, we have a simplification of interest:

$$(1.6)\qquad \sum_{n=0}^{\infty} \frac{t^n}{(1+an)} L_n^{v+an+avn}[x(1+an)] = (1-z)^{-1-v}\exp[xz/(z-1)],$$

where $t = z(1-z)^{a+av}\exp[axz/(1-z)]$ and $|t| < 1$. Letting $a \longrightarrow 0$ in (1.5) gives the known Carlitz expansion (1.1).

A third class of generating functions may be deduced from Theorem 2, which assumes the form

$$(1.7)\qquad \begin{aligned} &\sum_{n=0}^{\infty} \frac{t^n}{(n+1)} L_n^{v+bn}[x(1+an)] \\ &= \frac{\exp[(-xz+x-ax)/(1-z)]}{(1-z)^v z(v-b)} \left\{ {}_1F_1\left[\begin{matrix} v-b; & \dfrac{x(a-1)}{1-z} \\ v-b+1; & \end{matrix}\right] \right. \\ &\qquad \left. - (1-z)^{v-b}\,{}_1F_1\left[\begin{matrix} v-b; & x(a-1) \\ v-b+1; & \end{matrix}\right] \right\}, \end{aligned}$$

where $t = z(1-z)^b\exp[axz/(1-z)]$ and $|t| < 1$.

The three classes of generating functions given above are in fact particular examples of the more general expansion

$$\sum_{n=0}^{\infty} \frac{t^n L_n^{v+bn}[x(1+an)]}{(l-v-bn-n)((l-v+s-bn-n)/(s))_{l'}}$$

$$(1.8) \quad = \frac{(1-z)^{l-v}\exp[xz/(z-1)]}{l'!} \sum_{r=0}^{l'} \sum_{n=0}^{l+sr} \frac{(-z)^n(1-z)^{rs-n}(-l')_r(-l-rs)_n}{n!r!(l-v+rs-bn-n)}$$

$$\cdot\, {}_1F_1\left[\begin{matrix} 1; \\ (v-l+b+1+bn+n-rs)/(b+1); \end{matrix} \quad \frac{xz(b+1+al-av+ars)}{(1-z)(b+1)}\right],$$

where $t = z(1-z)^b\exp[axz/(1-z)]$, v, a, b are arbitrary complex numbers, l, l', s nonnegative integers, and $|t| < 1$. Note that we have assumed l, l', and s to be nonnegative integers to terminate the series on the right-hand side of (1.8). This condition is not necessary. The result is valid provided both sides of (1.8) exist for arbitrary values of the parameters involved.

Letting $a = 0$ and $l' = 0$ in (1.8) gives essentially Eq. (3.4) of Zeitlin [10].

As the Bessel polynomial is expressed as [6, Volume I, p. 195] and the Poisson-Charlier polynomial as [6, Volume I, p. 268], then the formulae given for the Laguerre polynomial may be converted to generate the above-mentioned polynomials.

Theorems 3 and 4 involve the Hermite and generalized Hermite polynomials. A special case is the generating function

$$(1.9) \quad \sum_{n=0}^{\infty} \frac{(1+bn)^{\frac{1}{2}n}}{n!} t^n H_n[x(1+an)/(1+bn)^{\frac{1}{2}}] = \exp[-z^2-2xz]\,[1+2bz^2+2axz]^{-1}$$

where $t = (-z)\exp[bz^2 + 2axz]$ and $|2axz\exp[bz^2 + 2axz + 1]| < 1$. The Hermite polynomial is defined in [6, Volume II, p. 193]. The special case $a = b = 0$ in (1.9) gives the old and well-known result [6, Volume II, p. 194, Eq. (19)]. If $b = 0$ and $a = 0$ in (1.9), then the equations (2.14) and (2.16), respectively, of Cohen [4], present themselves.

From Theorem 4, one class of new generating functions that may be deduced is

$$(1.10) \quad \sum_{n=0}^{\infty} \frac{t^n(1+bn)^{\frac{1}{2}(n-2)}}{n!} H_n[x(1+an)/(1+bn)^{\frac{1}{2}}] = \exp[-z^2-2xz]\,{}_1F_1\left[\begin{matrix} 1; \\ \frac{b+1}{b}; \end{matrix} \quad 2xz(b-a)/b\right],$$

where $t = (-z)\exp(bz^2 + 2axz)$ and $|2axz\exp(bz^2 + 2axz + 1)| < 1$. If $b \to 0$ and $a \to 0$ in (1.10), then Eqs. (2.14) and (2.17), respectively, of [4] are obtained. A special case of interest may be derived from (1.10) for $a = b$ to give

$$(1.11)\qquad \sum_{n=0}^{\infty} \frac{t^n(1+an)^{\frac{1}{2}(n-2)}}{n!} H_n[x(1+an)^{\frac{1}{2}}] = \exp[-z^2-2xz],$$

where $t = (-z)\exp(az^2 + 2axz)$ and $|2axz\exp[az^2 + 2axz + 1]| < 1$.

A second class of generating functions is a special case of Theorem 4, and may be expressed as

$$(1.12)\qquad \sum_{n=0}^{\infty} \frac{t^n(1+bn)^{\frac{1}{2}n}}{n!(1+an)} H_n[x(1+an)/(1+bn)^{\frac{1}{2}}] = \exp[-z^2-2xz]\,{}_1F_1\left[\begin{matrix}1;\\ \frac{1+2a}{2a};\end{matrix}\ z^2(a-b)/a\right],$$

where $t = (-z)\exp[bz^2 + 2axz]$ and $|2axz\exp[bz^2 + 2axz + 1]| < 1$. Letting $b \longrightarrow 0$ and $a \longrightarrow 0$ in (1.12) give the known Eqs. (2.15) and (2.16), respectively, of [4].

2. Generating Functions. The method of proof in the four theorems is a modification and extension of the one employed in obtaining new generating functions for the Jacobi polynomial in [5]. As in [5], no appeal is made to the Lagrange Theorem. It differs from that of previous workers, including the approach in [4].

THEOREM 1. *For α, β, s arbitrary complex numbers and r a positive integer*

$$(2.1)\qquad \sum_{k=0}^{\infty} \frac{t^k(\alpha+1+sk)_k}{k!}\,{}_rF_r\left[\begin{matrix}\Delta(-k,r);\\ \Delta(-\alpha-sk-k,r);\end{matrix}\ x(\beta+sk)\right] = (1-z)^{\alpha+1}(1+sz+rsy)^{-1}\exp[-\beta y],$$

where $t = (-z)(1-z)^{-s-1}\exp(sy)$, $x = (-y)[(z-1)/z]^r$, $(a)_k = \Gamma(a+k)/\Gamma(a)$, quotient of gamma functions $= a(a+1)(a+2)\ldots(a+k-1)$ for k a positive integer, and $\Delta(-k,r) = -k/r, (-k+1)/r, \ldots, (-k+r-1)/r$. $|t| < 1$, and ${}_rF_r$ is the generalized hypergeometric polynomial [9, *p.* 155].

Proof. Let us begin with the expression

$$(2.2)\qquad \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{z^n y^m}{n!m!} D^n[x^{\alpha-\beta+n}\delta^m\{x^\beta(1-x^s)^n(1-x^{s'})^m\}],$$

where $D \equiv d/dx$ and $\delta \equiv x\,d/dx$. Now

$$(2.3)\qquad D^n[x^{\alpha-\beta+n}\delta^m\{x^\beta(1-x^s)^n(1-x^{s'})^m\}]$$

$$(2.4)\qquad = \sum_{k=0}^{n}\sum_{p=0}^{m} \frac{(-n)_k(-m)_p(\beta+sk+s'p)^m(\alpha+1+sk+s'p)_n x^{\alpha+sk+s'p}}{k!p!}.$$

Taking (2.3) and (2.4) for $x = 1$, and putting the results in (2.2), one has

(2.5) $$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{m}\sum_{k=0}^{n}\frac{z^n y^m(-n)_k(-m)_p(\beta+sk+s'p)^m(\alpha+1+sk+s'p)_n}{k!p!n!m!}$$

(2.6) $$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-sz)^n(-s'y)^m(m+n)!}{n!m!}$$

(2.7) $$=[1+sz+s'y]^{-1}.$$

We go from (2.6) to (2.7) using the binomial expansions. Now applying the series transformation

(2.8) $$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{m}\sum_{k=0}^{n} f(n, m, k, p) = \sum_{n,m,k,p=0}^{\infty} f(n+k, m+p, k, p)$$

to (2.5), and simplifying, gives

(2.9) $$\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-z)^k(-y)^p\exp[y(\beta+sk+s'p)]\,(\beta+sk+s'p)^p(\alpha+1+sk+s'p)_k}{k!p!(1-z)^{\alpha+1+sk+k+s'p}}.$$

The summation of the series over n and m is achieved with the aid of the binomial and exponential expansions. Equating (2.7) and (2.9), and employing

(2.10) $$\sum_{p=0}^{\infty}\sum_{k=0}^{\infty} f(p, k) = \sum_{k=0}^{\infty}\sum_{p=0}^{[r/k]} f(p, k-rp)$$

with $r = s'/s$, and some algebra, gives the required theorem.

THEOREM 2. *For α, β, s, s' arbitrary complex numbers, l, l', s'' nonnegative integers and r a positive integer*

$$\sum_{k=0}^{\infty}\frac{(\alpha+1+sk)_k t^k}{(\alpha+l+1+sk)k!((\alpha+l+1+s''+sk)/s'')_{l'}}$$

$$\cdot\,{}_rF_r\left[\begin{matrix}\Delta(-k, r); \\ \Delta(-\alpha-sk-k, r);\end{matrix}\quad x(\beta+sk)\right]$$

(2.11)
$$=\frac{(1-z)^{\alpha+l+1}\exp(-\beta y)}{l'!}\cdot\sum_{q=0}^{l'}\sum_{k=0}^{l+s''q}\frac{(-z)^k(1-z)^{s''q-k}(-l')_q(-l-s''q)_k}{k!q!(\alpha+l+1+s''q+sk)}$$

$$\cdot\,{}_1F_1\left[\begin{matrix}1; \\ (\alpha+l+1+s'+s''r+sk)/s';\end{matrix}\quad y(\beta-\alpha-l-1-s''q)\right],$$

where $t = (-z)(1-z)^{-s-1}\exp(sy)$, $x = (-y)[(z-1)/z]^r$, and $|t| < 1$. Other symbols are defined in Theorem 1.

Proof. We first evaluate the integral

$$\int_0^1 x^l(1-x^{s''})^{l'} D^n [x^{\alpha-\beta+n}\delta^m \{x^\beta(1-x^s)^n(1-x^{s'})^m\}]\, dx. \tag{2.12}$$

Expanding the operators as in Theorem 1, we have

$$\sum_{p=0}^{m}\sum_{k=0}^{n} \frac{(-n)_k(-m)_p(\beta+sk+s'p)^m(\alpha+1+sk+s'p)_n}{k!p!} \cdot \int_0^1 x^{sk+s'p+l}(1-x^{s''})^{l'}\, dx. \tag{2.13}$$

The integral in (2.13) may be evaluated to give

$$\frac{l'!}{(\alpha+l+1+sk+s'p)((\alpha+l+1+s''+sk+s'p)/s'')_{l'}}. \tag{2.14}$$

Going back to (2.12), and expanding $(1-x^{s''})^{l'}$ gives

$$\sum_{q=0}^{l'} \frac{(-l')_q}{q!} \int_0^1 x^{l+s''q} D^n [x^{\alpha-\beta+n}\delta^m \{x^\beta(1-x^s)^n(1-x^{s'})^m\}]\, dx. \tag{2.15}$$

Now integrating by parts n times, then expanding $(1-x^s)^n$ and $(1-x^{s'})^m$ and integrating again, (2.15) reduces to

$$\sum_{q=0}^{l'}\sum_{p=0}^{m}\sum_{k=0}^{n} \frac{(-l')_q(-l-s''q)_n(-n)_k(-m)_p(\beta+sk+s'p)^m}{q!p!k!(\alpha+l+1+s''q+sk+s'p)}. \tag{2.16}$$

Equating (2.13) and (2.14) to (2.16), multiplying both sides of the resulting equation by

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{z^n y^m}{n!m!}$$

and proceeding as in Theorem 1, we have, after simplification

$$\sum_{p=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-z)^k(-y)^p \exp[y(\beta+sk+s'p)](\beta+sk+s'p)^p(\alpha+1+sk+s'p)_k}{k!p!(1-z)^{\alpha+1+sk+k+s'p}(\alpha+1+l+sk+s'p)((\alpha+l+s''+1+sk+s'p)/s'')_{l'}}$$

$$= \frac{1}{l'!}\sum_{q=0}^{l'}\sum_{k=0}^{l+s''q} \frac{(-z)^k(1-z)^{l+s''q-k}(-l')_q(-l-s''q)_k}{k!q!(\alpha+l+1+s''q+sk)} \cdot {}_1F_1\left[\begin{matrix}1; \\ (\alpha+l+s'+1+s''q+sk)/s';\end{matrix}\; y(\beta-\alpha-l-1-s''q)\right]. \tag{2.17}$$

Applying the series transformation (2.10) and simplifying, gives Theorem 2.

THEOREM 3. *For α β, s' arbitrary complex numbers and r a positive integer*

$$\sum_{p=0}^{\infty} \frac{t^p(\beta+s'p)^p}{p!}\, {}_rF_0\left[\Delta(-p,r);\, \text{——};\, \frac{xr^r(\alpha+s'p)}{(\beta+s'p)^r}\right]$$

$$= \exp(-\beta y-\alpha z)(1+s'y+rs'z)^{-1}, \tag{2.18}$$

where $t = (-y)\exp(s'y + s'z)$, $x = (-z)(y)^{-r}$, *and* $|s'y\exp(s'y + s'z + 1)| < 1$. *See Theorem* 1 *for other symbols.*

Proof. Consider

$$\sum_{m=0}^{\infty}\sum_{n=0}^{n}\frac{z^n y^m}{n!m!}\,\delta^n[x^{\alpha-\beta}\delta^m\{x^{\beta}(1-x^{s})^n(1-x^{s'})^m\}]. \tag{2.19}$$

At $x = 1$ we have, after expansion

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z^n y^m}{n!m!}\sum_{p=0}^{m}\sum_{k=0}^{n}\frac{(-n)_k(-m)_p(\alpha+sk+s'p)^n(\beta+sk+s'p)^m}{k!p!}. \tag{2.20}$$

But at $x = 1$, (2.19) also reduces to

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-sz)^n(-s'y)^m(n+m)!}{n!m!}. \tag{2.21}$$

Equating (2.20) to (2.21), and proceeding as in Theorem 1, with applying series transformations and simplifying, gives

$$\sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-z)^k(-y)^p\exp[y(\beta+sk+s'p)+z(\alpha+sk+s'p)](\alpha+sk+s'p)^k(\beta+sk+s'p)^p}{k!p!} = (1+zs+ys')^{-1}. \tag{2.22}$$

Using the series transformation (2.10) and some manipulation results in Theorem 3. Note that for $s' = 0$ in the above theorem, the ${}_rF_0$ hypergeometric polynomial is essentially the generalized Hermite polynomial which occurs in probability problems. It has been studied by Gould and Hopper [7], Gupta and Jain [8], Cohen [4], and others.

THEOREM 4. *For* α, β, s, s' *arbitrary complex numbers and* r, r' *positive integers*

(a)

$$\sum_{p=0}^{\infty}\frac{(\beta+s'p)^p t^p}{p!(\alpha+s'p)}\,{}_rF_0\left[\Delta(-p,r);\,\text{—};\,\frac{xr^r(\alpha+s'p)}{(\beta+s'p)^r}\right] = \frac{\exp(-\alpha z-\beta y)}{(\alpha)}\,{}_1F_1\left[\begin{matrix}1;\\(\alpha+s')/s';\end{matrix}\;y(\beta-\alpha)\right], \tag{2.23}$$

where $t = (-y)\exp(s'y + s'z)$, $x = (-z)(y)^{-r}$, *and* $|s'y\exp(s'y + s'z + 1)| < 1$.

(b)

$$\sum_{k=0}^{\infty}\frac{(\alpha+sk)^{k-1}t'^k}{k!}\,{}_{r'}F_0\left[\Delta(-k,r');\,\text{—};\,\frac{r'^{r'}x'(\beta+sk)}{(\alpha+sk)^{r'}}\right] = \frac{\exp(-\alpha z-\beta y)}{(\alpha)}\,{}_1F_1\left[\begin{matrix}1;\\(\alpha+r's)/r's;\end{matrix}\;y(\beta-\alpha)\right] \tag{2.24}$$

where $t' = (-z)\exp(sy + sz)$, $x' = (-y)(z)^{-r'}$, *and* $|sz\exp(sy + sz + 1)| < 1$. *Other symbols are defined in Theorem* 1.

Proof. Let us evaluate the expression

$$(2.25)\qquad \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z^n y^m}{n!m!}\int_0^1 x^{-1}\delta^n[x^{\alpha-\beta}\delta^m\{x^\beta(1-x^s)^n(1-x^{s'})^m\}]\,dx.$$

Expanding and operating as in the previous theorems, we have

$$(2.26)\qquad \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z^n y^m}{n!m!}\sum_{p=0}^{m}\sum_{k=0}^{n}\frac{(-n)_k(-m)_p(\beta+sk+s'p)^m(\alpha+sk+s'p)^{n-1}}{k!p!}.$$

Going back to (2.25) and integrating, we find the only contribution comes from $n = 0$ and hence (2.25) may be evaluated to give

$$(2.27)\qquad \sum_{m=0}^{\infty}\frac{[y(\beta-\alpha)]^m m!}{m!(\alpha)((\alpha+s')/s')_m}.$$

Equating (2.26) and (2.27), using the series transformation, and reducing the summation over n and m, one obtains the expression

$$\sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-z)^k(-y)^p(\beta+sk+s'p)^p(\alpha+sk+s'p)^{k-1}\exp[z(\alpha+sk+s'p)+y(\beta+sk+s'p)]}{k!p!}$$

$$(2.28)\qquad = (1/\alpha)\,{}_1F_1\left[\begin{matrix}1; \\ (\alpha+s')/s';\end{matrix}\; y(\beta-\alpha)\right].$$

Now applying (2.10) to (2.28) twice, with p and k interchanged, and some algebra, gives the two parts of the theorem. Note that $r = s/s'$, and $r' = s'/s$.

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